

# Neural ODEs with stochastic vector field mixtures

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**Abstract.** It was recently shown that neural ordinary differential equation models cannot solve fundamental and seemingly straightforward tasks even with high-capacity vector field representations. This paper introduces two other fundamental tasks to the set that baseline methods cannot solve, and proposes mixtures of stochastic vector fields as a model class that is capable of solving these essential problems. Dynamic vector field selection is of critical importance for our model, and our approach is to propagate component uncertainty over the integration interval with a technique based on forward filtering. We also formalise several loss functions that encourage desirable properties on the trajectory paths, and of particular interest are those that directly encourage fewer expected function evaluations. Experimentally, we demonstrate that our model class is capable of capturing the natural dynamics of human behaviour; a notoriously volatile application area. Baseline approaches cannot model this problem.

## 1 Introduction

Recent work has linked Residual Networks (ResNets) [7] to Ordinary Differential Equations (ODEs) and demonstrated that ResNets may be interpreted as Euler solutions to Initial Value Problems (IVPs) [12, 22, 17]. This idea has been further developed with ResNets taken to their continuous limit by [3] with the introduction of Neural ODEs (NODEs). In their setting, Neural Networks (NNs) produce Vector Field (VF) representations that are utilised by ODE solvers to learn and calculate explicit flow trajectories through the feature space. The resulting model can be seen to be of continuous depth since the ODE solver will query the VFs at arbitrary levels that are unspecified in advance. Their work also demonstrates efficient and scalable inference with the adjoint trick.

Vector fields are defined as follows in the NODE paradigm

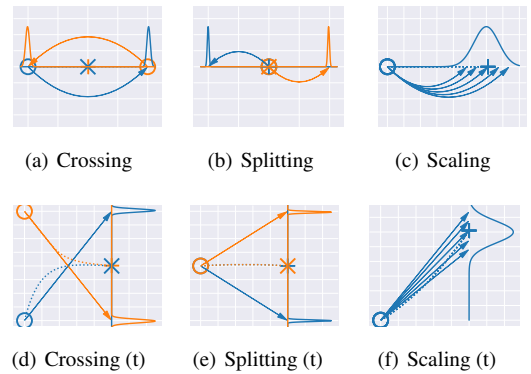
$$\nabla \mathbf{h}(t_i) = f(\mathbf{h}(t_i), t_i; \boldsymbol{\theta}) \quad (1)$$

where  $\mathbf{h}(t_i) \in \mathbb{R}^D$  is a hidden state at depth  $t_i$  (base state  $\mathbf{h}(t_0) \triangleq \mathbf{x}$ , maximal depth  $t_T$ ),  $f$  is a neural network of arbitrary architecture and  $\boldsymbol{\theta}$  are its parameters. Subsequent hidden states are derived by taking small steps from  $\mathbf{h}(t_i)$  in the direction of the vector field, *i.e.*  $\mathbf{h}(t_{i+1}) = \mathbf{h}(t_i) + \delta t_i \nabla \mathbf{h}(t_i)$ , and the step size  $\delta t_i$  is often selected automatically by an ODE solver. The solution of this IVP is obtained by repeatedly taking these steps until the output state  $\mathbf{h}(t_T)$  is reached, and these outputs may undergo a final transformation, *e.g.* through a softmax layer in classification tasks. The optimisation objective is to adjust the dynamics of the ODE through  $\boldsymbol{\theta}$  to maximise the data likelihood.

A critical component of NODEs is the parametrisation of the VFs and, since  $f$  is a neural network, the practitioner has the important responsibility of appropriately specifying its architecture. Even though the VF representations will be non-linear (and arbitrarily ‘flexible’ depending on architecture choice), NODE models cannot model arbitrary problems. This was clearly demonstrated by [5] with a seemingly straightforward task in one dimension. A simple and elegant extension called Augmented NODEs (ANODEs) is proposed. These are experimentally shown to be a more expressive and stable model choice capable of solving the crossing problem.

Figure 1 illustrates NODE failure cases. Crossing [5] tasks cannot be solved by NODEs since their advections constitute homomorphisms for which crossing is forbidden. However, ANODE models find solutions by lifting the representation out of the original feature domain. Additionally, neither NODE nor ANODE models can solve splitting problems (where a single datapoint maps to more than one target) or account for uncertainty and hence they always map to a single target regardless of the variance of the system (*c.f.* scaling cases in the figure).

The shortcomings of baseline approaches are addressed with a probabilistic approach to ODE problems in this work. We use indoor localisation as a motivating application which seeks to find a mapping to a relative coordinate system. We propose the use of NODE-like models to learn typical patterns of movement and later to deploy these to forecast future behaviour. The three cases in fig. 1 are often encountered in behavioural modelling as follows: crossing: residents will never take the exact same path between two locations, and exemplar paths are likely to cross at least once; splitting: residents have the



**Figure 1.** This figure illustrates failure cases for NODE models in their original space (left) and particle evolution over the integration interval (right). In these figures colour indicates class membership, the solid arrows link the desired start (o) and end (x +) positions of particles (*i.e.* ground truth data). Dotted lines depict the learnt transportation by NODE models. The Gaussian PDFs represent the predictive distributions of the proposed method.

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option to turn left, right, move forward or backtrack on their trajectory with their latent intent behind their motion determining this decision; and scaling: unobservable factors influence paths walked by residents, and even under identical starting conditions a resident may move at different speeds/directions along the same path.

Our main contribution is the introduction and exposition of Stochastic VF Mixtures (SVFMs) within the NODE paradigm, and the introduction of trajectory-focused losses that encourage simpler and less distorted solutions. Their utility is primarily enabled by the uncertainty that is posited on both the vector fields themselves and on selection of vector fields from an ensemble, but is also supported by loss functions that encourage simpler solutions to the IVP and capture richer context in forecasting tasks. We demonstrate their capability in solving the above three fundamental tasks where baseline models fail. Although the proposed methods are developed specifically with forecasting in mind, we also demonstrate their utility in classical classification tasks.

## 2 Proposed methods

This section outlines an approach for modelling VF mixtures and stochastic VFs. Additionally, several loss functions are introduced that encourage desirable properties for our models.

### 2.1 Vector field mixtures

The graphical model representing the basic VF unit is shown in fig. 2(a). The input variables ( $t$  and  $\mathbf{h}(t)$ ) are denoted by shaded nodes indicating that these values are observed, whereas the output node ( $\nabla \mathbf{h}(t)$ ) is unshaded, indicating that it is latent. The assumption here is that the  $i$ -th VF is conditionally independent of its history given its hidden state. In what follows we use the notation  $f_g(\mathbf{h}(t), t; \theta^{(g)})$  to represent a NODE unit on an example latent variable  $g$ , whereas  $f^{(k)}(\cdot)$  specifically defines the  $k$ -th VF from the VF Mixture (VFM).

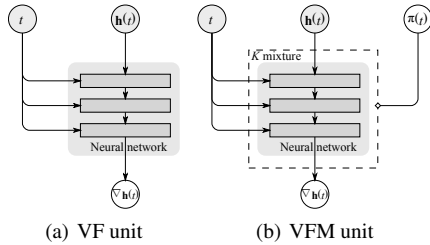


Figure 2. Architecture of VF and VFM units.

Using gate notation [13], a VFM with  $K$  components is shown in fig. 2(b). The new latent variable,  $\pi(t) \in \Delta^K$  (where  $\Delta^K$  defines the  $K$ -simplex) is a discrete probability distribution over component membership. The manner in which this distribution is specified is a key consideration with these models. Two approaches for defining  $\pi(t)$  and propagating membership belief over the integration interval are outlined below.

**Pick and stick.** This operates under the assumption that component membership is constant over the forecast horizon, *i.e.*  $\pi(t_i) = \pi(t_*) \forall i$  ( $0 \leq i \leq T$ ). Several choices for specifying  $t_*$  may be considered, but if mixture dynamics are well specified at  $t_0$  we can set

$$\pi(t_*) = \pi(t_0) = f_{\pi_{t_0}}(\mathbf{h}(t_0), t_0; \theta^{(\pi_{t_0})}) \quad (2)$$

**Forward filtering.** Forward filtering is a technique used in dynamic systems to estimate belief based on a history of evidence, and it is often used in dynamic models like Hidden Markov Models (HMMs) [15] and Conditional Random Fields (CRFs) [19]. It is incorporated into our model directly by modeling transition and emission dynamics of component membership as follows

$$\Psi(t_i) = f_{\Psi}(\mathbf{h}(t_i), t_i; \theta^{(\Psi)}) \quad (3)$$

$$\psi(t_i) = f_{\psi}(\mathbf{h}(t_i), t_i; \theta^{(\psi)}) \quad (4)$$

where  $\psi(t_i)$  and each of the  $K$  rows of  $\Psi(t_i) \in \Delta^K$ . Forward filtering aggregates belief as follows

$$\pi(t_i) \propto \Psi(t_i)^\top (\psi(t_i) \odot \pi(t_{i-1})) \quad (5)$$

where  $\odot$  depicts the Hadamard product,  $\top$  the matrix transpose, and equality is achieved with normalisation. A complementary interpretation of forward filtering is that of a Recurrent NN (RNN) [25] that grows probabilistically through the forecasting horizon.

### 2.2 Stochastic vector fields

Uncertainty is introduced to VFs with the mean ( $\mu(t) = g_{\mu}(f_{\mathbf{z}}(t), t; \theta^{(\mu)})$ ) and variance ( $\tau(t) = g_{\tau}(f_{\mathbf{z}}(t), t; \theta^{(\tau)})$ ) generating functions that both arise from a common representation  $f_{\mathbf{z}}(t) = g_{\mathbf{z}}(\mathbf{h}(t), t; \theta^{(z)})$ , see fig. 3. It will be convenient to de-

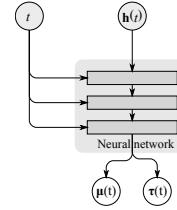


Figure 3. Architecture of the SVF unit.

compose VFs into the length ( $v(t)$ ) and orientation ( $\mathbf{u}(t)$ ) of the VF and to impose uncertainty on these separately:

$$\mathbf{u}(t) \sim \mathcal{N}_s(\mu^{(\mathbf{u})}(t), \tau^{(\mathbf{u})}(t)) \quad (6)$$

$$\log v(t) \sim \mathcal{N}(\log \mu^{(v)}(t), \tau^{(v)}(t)) \quad (7)$$

where  $\mu^{(\mathbf{u})}(\cdot)$ ,  $\tau^{(\mathbf{u})}(\cdot)$ ,  $\mu^{(v)}(\cdot)$  and  $\tau^{(v)}(\cdot)$  are the mean and variance representations of the direction and length variables respectively,  $\mathcal{N}(\cdot)$  is a Gaussian distribution and  $\mathcal{N}_s(\cdot)$  is a distribution defined on a  $D$ -sphere. Samples from both distributions can be transformed into VFs with the inverse transformation to that used above,  $\nabla \mathbf{h}(t) = \exp\{v(t)\} \mathbf{u}(t)$ .

In principle one may wish to constrain the variance of Stochastic VF (SVF) to ensure that  $\mathbf{u}(t) \cdot \mu^{(\mathbf{u})}(t) > 0$  since, with large variance, samples from the directional distribution (eq. (6)) will be reversed with respect to the expected direction of the VF. We believe this directional preservation property is important since the expected inductive bias of the SVF for each sample is continued. This tends to give lower variance solutions. A similar motivation led to us imposing uncertainty on length in the log domain. Although noise can be imposed on VFs without the decomposition (*e.g.* see [14]), the proposed decomposition naturally orientates the distribution along the expected direction of the VF.

## 2.3 Loss functions

The complexity of IVP solutions in the NODE framework is typically reported with the Number of Function Evaluations (NFE) metric. NFEs are influenced by the total distance and the ‘straightness’ of the solution trajectory, and here we introduce and motivate losses that encourage fewer NFEs. In so doing, we achieve solutions that preserve a significant amount of the original topology of the data (since the majority of the data remains stationary) but we also expose a link between NFE minimisation and the variance of VFs over the solution path.

### 2.3.1 NFE losses

The two losses discussed in this subsection directly relate to NFE reduction since they target short solution paths and low-variance VFs.

**Transportation Loss (TLoss)** TLoss is defined as follows

$$\mathcal{L}_{\text{TLoss}} = \frac{1}{T} \sum_{i=1}^T \|\mathbf{h}(t_i) - \mathbf{h}(t_{i-1})\|_2^2 \quad (8)$$

The minimal possible  $\mathcal{L}_{\text{TLoss}}$  between  $\mathbf{h}(t_0)$  and  $\mathbf{h}(t_T)$  is  $\|\mathbf{h}(t_T) - \mathbf{h}(t_0)\|_2^2$  and this is achieved when all VF evaluations point in the same direction, *i.e.*  $\nabla \mathbf{h}(t_i) = \nabla \mathbf{h}(t_0) s_i \forall s_i \in \mathbb{R}_+$ .

**Variance Loss (VLoss)** VFs can be encouraged to point in the same direction by regularising their variance. Letting  $\mathbb{E}[\nabla \mathbf{h}(t)] = \frac{1}{T} \sum_{i=1}^T \nabla \mathbf{h}(t_i)$  be the expected VF, VLoss is defined as

$$\mathcal{L}_{\text{VLoss}} = \frac{1}{T} \sum_{t=1}^T \|\nabla \mathbf{h}(t_i) - \mathbb{E}[\nabla \mathbf{h}(t)]\|_2^2 \quad (9)$$

Combined transportation distance and directional variance losses are termed Transport & Variance Loss (TVLoss).

### 2.3.2 General losses

The losses in this subsection are specified for the learning over mixture distributions and forecasting.

**Mixture Density Loss (MDLoss)** SVFM models produce mixture distribution as outputs, and these are reconciled into a mixture density loss as follows

$$\mathcal{L}_{\text{MDLoss}} = -\log \sum_{k=1}^K p\left(\mathbf{h}(t) | \boldsymbol{\mu}^{(k)}(t), \boldsymbol{\tau}^{(k)}(t)\right) \boldsymbol{\pi}^{(k)}(t) \quad (10)$$

where  $p$  is the density of the output distribution. MDLoss resembles those in mixture density networks [1].

**Forecasting Loss (FLoss)** In forecasting problems, a datapoint is a sequence of  $T'$  measurements and timestamps. We define  $\mathbf{X} \in \mathbb{R}^{T' \times D}$  to be the matrix of measurements and  $\mathbf{t}_X \in \mathbb{R}^{T'}$  to be the corresponding timestamps. Forecasting loss is then defined as

$$\mathcal{L}_{\text{FLoss}} = \frac{1}{T} \sum_{i=1}^T \ell(\mathbf{h}(t_i), \text{interp}(\mathbf{X}, \mathbf{t}_X, t_i)) \quad (11)$$

where  $\text{interp}$  is an interpolation function that estimates data at  $t_i$ , and  $\ell$  is a secondary loss that penalises  $\mathbf{h}(t_i)$  that are far from the interpolated target. We choose cubic interpolation in our experiments and the choice of  $\ell$  dependent on the model; mean squared error is used with baseline methods and MDLoss is used with the SVFMs.

### 2.3.3 The case for low-variance VFs

Aligned VFs are encouraged by the variance reduction loss discussed above. Here we expand on the computational gains achieved when solving IVPs with aligned VFs.

We formalise this analysis with Runge-Kutta-based iterative embedded ODE solvers. These can be characterised by their Butcher tableau [2] that defines the coefficients, temporal offsets and evaluation weighting of VFs. The Butcher tableau of an order  $s$  embedded solver is often tabulated as

$c_1$	$a_{11}$	$a_{12}$	$\dots$	$a_{1s}$
$c_2$	$a_{21}$	$a_{22}$	$\dots$	$a_{2s}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$c_s$	$a_{s1}$	$a_{s2}$	$\dots$	$a_{ss}$
	$b_1$	$b_2$	$\dots$	$b_s$
	$b_1^*$	$b_2^*$	$\dots$	$b_s^*$

and for a given state,  $\mathbf{h}(t_n)$ , the next output is given by

$$\mathbf{h}(t_{n+1}) = \mathbf{h}(t_n) + h \sum_{i=1}^s b_i \mathbf{k}_i \quad (12)$$

where the intermediate vector field evaluations are defined as

$$\mathbf{k}_i = f\left(\mathbf{h}(t_n) + \sum_{j=1}^s a_{ij} \mathbf{k}_j, t_n + c_i h\right) \quad (13)$$

and where  $a$ ,  $b$  and  $c$  are as defined in the tableau above,  $h$  is the step size for that iteration, and  $f$  is defined in eq. (1). The final row of the tableau is used to estimate truncation error as follows

$$\mathbf{h}^*(t_{n+1}) = \mathbf{h}(t_n) + h \sum_{i=1}^s b_i^* \mathbf{k}_i \quad (14)$$

$$\begin{aligned} e_{t_{n+1}} &= \mathbf{h}(t_{n+1}) - \mathbf{h}^*(t_{n+1}) \\ &= h \sum_{i=1}^s (b_i - b_i^*) \mathbf{k}_i \end{aligned} \quad (15)$$

The step size  $h$  is adjusted dynamically when deriving the solution to be close to a specified tolerance value. If the error is too high,  $h$  is reduced and the iteration is repeated. If the error below the tolerance,  $h$  is accepted but the step size may be increased for the next iteration.

If VLoss is incorporated and successfully aligns all VFs then each can be seen as a scaled version of the principal direction,  $\bar{\mathbf{k}}$ , *i.e.*

$$e(t_{n+1}) = h \bar{\mathbf{k}} \sum_{i=1}^s (b_i - b_i^*) \quad (16)$$

Notice that the truncation error has become independent of the intermediate evaluation points with the alignment. By design  $\sum_i b_i = 1$  and  $\sum_i b_i^* = 1$  and so the truncation error tends to 0 in the equation above. This allows solutions to be discovered with just one step. Although the systems characterised by eq. (16) are not particularly interesting in general, we introduce it since it is minimal in the sense of NFE requirements. In other words, for a given tolerance, larger step sizes can be afforded with aligned VFs. Our experiments demonstrate these savings clearly when these losses have been incorporated, which can be seen to tend towards this limiting case. Interestingly, this suggests a link between transportation and variance losses since under the conditions of eq. (16) both are minimal.

### 3 Sampling and SVFM relationships

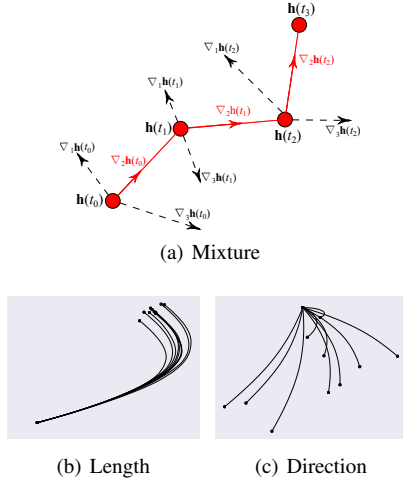


Figure 4. Illustration of three forms of uncertainty.

Figure 4(a) illustrates branching patterns that are characteristic of VFM solutions. The initial state  $\mathbf{h}(t_0)$  is shown by the red node on the bottom left, and its trajectory to  $\mathbf{h}(t_3)$  is highlighted in red. At each evaluation position one of three VFs is dynamically sampled from the distribution  $\pi(t_i)$ . Although we only consider myopic approaches here, *i.e.* steps are taken based on samples from  $\pi(t_i)$ , other approaches (including particle filtering and reinforcement learning) will be explored in future work. Exhaustive search is prohibitive since it is exponential on the number of evaluation points. Ten samples from uncertain VFs are shown in figs. 4(b) and 4(c). Scaled VFs (fig. 4(b)) which always initially follow the same trajectory will tend to ‘fan out’ as curvature is encountered whereas directional uncertainty (fig. 4(c)) tends to fan out immediately. The full factor graph of the proposed approach is shown in fig. 5. NODE units are depicted by the black squares in this figure.

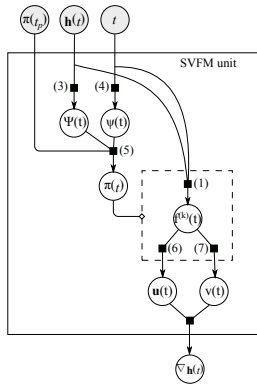


Figure 5. Full architecture of the SVFM unit.

The proposed model, SVFM, encapsulates all of the properties shown in fig. 5. Sequences of samples from SVFMs are likely to vary significantly even under the same contextual conditions, and this will cause problems for ODE solvers. Since the classic Runge-Kutta 4 (RK4) method requires four function evaluations in determining the next direction [11] and the Dormand-Prince (DP) method requires six [4] their convergence guarantees do not hold since SVFMs are

discontinuous everywhere [18], but with careful scheduling it is possible for these guarantees to hold at each step. Our approach is to record the random state at the beginning of the evaluation campaign, and to reset the random number generator’s state to the recorded state whenever samples need to be drawn during the campaign, and these are used with the reparameterisation trick on Gaussian and discrete distributions [16, 9]. Experimentally, this trick significantly reduces the number of function evaluations required with DP solvers that are used in our experiments. The incorporation of uncertainty and mixture distributions to VFs are the key factors that help solve scaling and splitting cases and also facilitate behavioural modelling. Figure 6 highlights the relationship between stochastic (S-, blue), mixture (-M, black) and augmented (A-, red) [5] models. This figure not only shows that these three modelling choices can be used together, but that any combination can be selected.

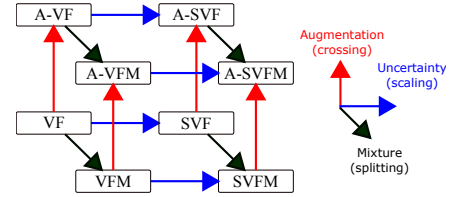


Figure 6. Relationships between discussed models.

Although the most complete model is A-SVFM as it incorporates all three components, our experimentation will primarily be evaluated with SVFM models because our interest is in forecasting and augmentation lifts the representation out of the original feature space.

Stochastic VFs have also been studied principally within the Gaussian Process (GP) [23] framework [24, 21]. Such models are non-parametric and require specification of covariance (kernel) functions. During inference, the covariance matrix underpinning the model requires inversion, which is a bottleneck for larger datasets. However, sparse methods for approximating the covariance with a subset of induced data [6] can trades computational cost off for approximation of the solution.

We investigated the properties of SVFM models theoretically. First we prove that NODE and ANODE models are unable to model splitting and scaling problems. We believe this further justifies our work since these problems are basic and (conceptually) simple components. Secondly we characterise the nature of VFs that require minimal NFEs. Although it turns out that the resulting VFs are trivial and are not useful in general, the analysis provides justification for the variance and transportation losses mentioned in the previous section.

## 4 Results and discussion

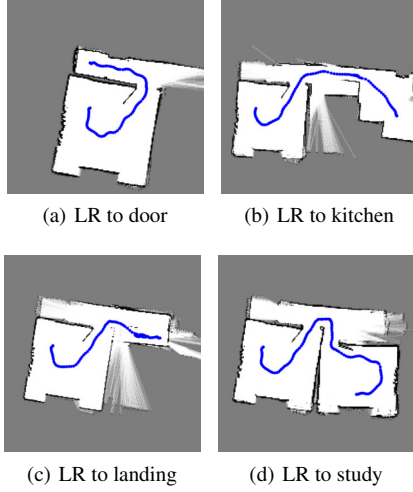
### 4.1 Experiments

#### 4.1.1 Synthetic data

Three example datasets are used in this paper: moons, nested circles and XOR. In all cases 1,000 datapoints are sampled for training, testing and validation.

#### 4.1.2 Behavioural data collection and processing

The behavioural dataset was collected in an adapted residential house. The collection mechanism involved a bespoke robotic wearable, capable of gathering 2-dimensional LiDAR point clouds from a human participant. The experimental procedure asked the participant to walk



**Figure 7.** Visualisation of SLAM maps mapping a single trajectory from the living room to four endpoints. We can see that the map derived from each setting is a faithful representation of the floorplan in fig. 12.

naturally into 4 different sections of the house from the same origin point, located in the living room. The 4 different targets in the house included the front door, the study room, the kitchen, and the landing of the stairs. Time of each experiment was assured through NTP synchronisation between the wearable and university NTP servers.

The procedure for collection followed a protocol:

1. Begin in the origin ‘anchor’.
2. Choose one of the four target locations.
3. Proceed to the target. This should take roughly 5-10 seconds.
4. Repeat 10 times.

The subsequent processing of the point cloud was performed using 2-dimensional SLAM. The data was associated using a loop closure method outlined in [8]. The map and the locations were extracted using an occupancy grid map [20]. The loop closure in this context specifies a method of location error minimisation from aggregated sub maps [8]. The parameters of the algorithm included laser reflectivity, internal loop closure threshold, resolution of the grid map and down-sampling of the data.

#### 4.1.3 Models

Several architectural, grid search and optimisation parameters used in our experiments are enumerated below:

- VF representer
  - Number of hidden layers: 1 and 2.
  - Number of hidden units: 32 and 64.
  - Activation function: rectified units in all cases.
  - Output VF activation: linear units.
- Optimisation
  - Optimizer: Adam [10].
  - Batch size: 50 and 100.
  - Learning rate:  $10^{-2}$ ,  $10^{-3}$ ,  $10^{-4}$ .
- ODE solver

- Tolerance:  $10^{-6}$ .

- SVFM

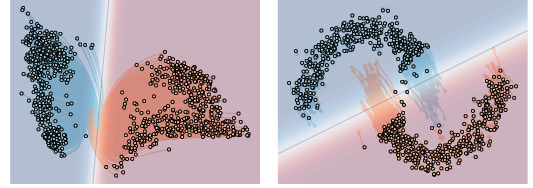
- Component selection: pick and stick, forward filtering.
- K from  $\{2, 4, 8\}$

We introduced four loss functions in the main paper: Mixture Density Loss (MDLoss), Transportation Loss (TLoss), VLoss and Forecasting Loss (FLoss). Since FLoss penalises deviation from a path, it is incompatible with TLoss and VLoss. Classification tasks, however, may mix TLoss and VLoss losses with predictive losses (*e.g.* cross-entropy). The balance between the predictive loss and the path regularisation is established with one parameter  $\lambda$ , and this is selected on the validation set.

## 4.2 Illustrative examples.

We provide several animated GIFs that illustrate the transformations that are described statically here<sup>4</sup>.

Figure 8 depicts the transforms that are learnt on the ‘moons’ dataset (*i.e.* two intersecting semi-circles of different classes). On the left are the transforms that are achieved by the baseline NODE model, but on the right the transforms achieved by the proposed SVFM model are shown. The problem is essentially solved in both cases, but the SVFM model has required minimal datapoint transformation to solve the task. We believe this characteristic illustrates advantageous traits in our task since in real applications, the domain knowledge of experts is still relevant in the majority of cases.



**Figure 8.** Moons predictions from NODE (left) and SVFM (right).

Figure 9 depicts transformation of VF (TR), VF with TVLoss (LL; herein called VF-TVLoss) and SVFM (LR) models on the nested circles dataset (*i.e.* samples from two circles of different radius each of a different class, see TL). The NODE solution has learnt a complex trajectory that requires the inner circle to ‘break’ through the outer circle. We note that particles are transported large distances through many NFE to break through, and this jeopardises generalisation accuracy. We can reduce NFE by incorporating TVLoss. Their effect is visibly seen on the top left of the LL figure where several blue points remain stationary at their original positions and all red trajectories are close to linear. However, in solving this problem many blue datapoints have been transported *around* the outer circle and these nonlinear trajectories require many NFEs. SVFM (LR) can be seen to produce linear transformations since the trajectory of any two particles are independent if assigned to different VF components.

The example in fig. 9 (LR) clearly demonstrates that SVFM models facilitate linear trajectories needing fewer function evaluations. We noted that for some simple toy datasets SVFM can learn a trivial solution by assigning  $\pi(t_0)$  directly associated to the class label. If

<sup>4</sup> Moons with VF; Moons with TVLoss; Nested circles with VF; Nested circles with TVLoss; and Nested circles with SVFM



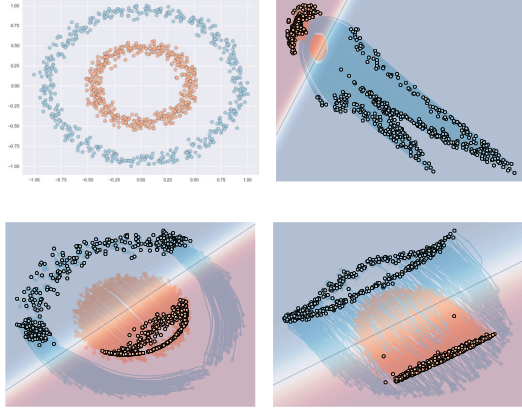


Figure 9. Nested circle predictions.

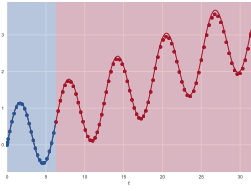


Figure 10. Cyclic continuation.

desired, this can be encouraged by penalising low mutual information between  $\pi$  and the target labels. We do not study this.

The integration variable  $t$  and the interval over which it is integrated are abstract quantities that directly affect the depth of the network. In our behavioural application, we interpret  $t$  as the Time of Day (ToD) and instead of parameterising our VF networks with the time variable  $t$  we use a time tuple  $\mathbf{t} = (\cos(t/l), \sin(t/l))$  ( $l = \frac{\pi}{12}$  if  $t$  is given in hours) since it has smooth and periodic characteristics for increasing  $t$ .

We call the augmentation of  $t$  to  $\mathbf{t}$  cyclic VF continuation and an example of it is illustrated in fig. 10. Here, the function  $g(t) = \sin(t) + t/10$  (shown in the solid trace) is learnt by a NODE mode with data from the blue interval ( $0 \leq t \leq 2\pi$ ). The model is then tested by querying the endpoint at  $t^* = 10\pi$ , a point well outside the training domain. When determining the endpoint, the NODE model performs several function evaluations shown with the blue and red dots. We see that the periodic traits of the function have been extrapolated well beyond the training interval into the red region. Of particular note is that all intermediate datapoint evaluations are faithful to the true function which is directly encouraged by the forecasting loss. In the context of behavioural modelling, this simple example demonstrates that cyclic VF continuation should allow for periodic behavioural traits to be representatively modelled over long forecast horizons. Although the demonstration here is rather straightforward, richer notions of meaningful periodic dependency will be captured from real datasets and tasks. Traditional VF parameterisations cannot learn even this basic extrapolation task.

### 4.3 Forward evaluation analysis

The effective complexity of NODE models is widely reported with the NFE metric, but NFE is a somewhat nebulous term since it more accurately captures the maximal NFE of a batch. We illustrate this by solving the IVP for every instance of a dataset separately and

producing a histogram on the set of NFEs. Taking the particular example of the VF model on the moons dataset (*i.e.* blue histogram in fig. 11(a)) the NFE metric (in the traditional sense) is 26, but it is clear that most datapoints require much fewer function evaluations, with the mean and median  $\approx 17$ . Computational savings can be achieved by separating the ‘hard’ and ‘easy’ instances from one another since it is the hardest instance of a batch that subjects the rest to unnecessary evaluations. Estimation error given by embedded solvers allow for dynamic evaluation.

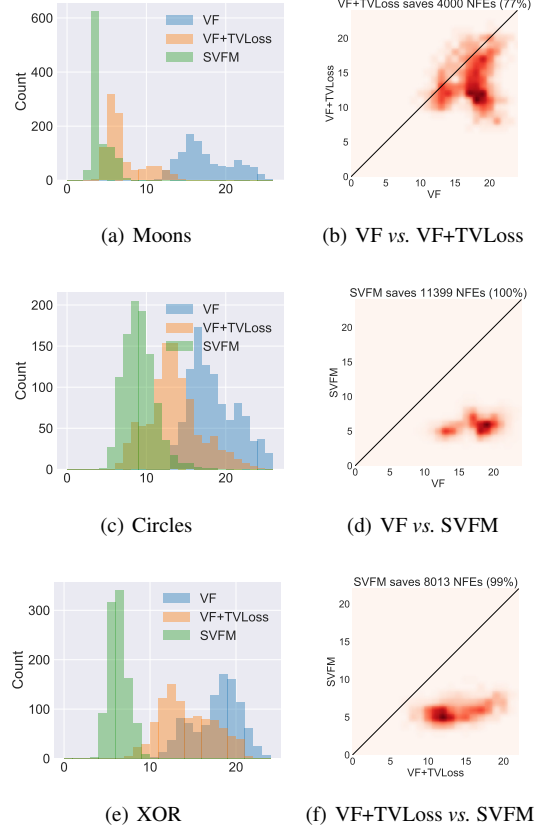


Figure 11. NFE histograms for baseline and proposed models on moons (fig. 11(a)), circles (fig. 11(c)) and XOR (fig. 11(e)) dataset. The per-instance NFE heatmap is shown for the XOR dataset in figs. 11(b), 11(d) and 11(f). Here, red indicates regions high density and the leading diagonal is the line of equal NFEs. SVFM models consistently require fewer NFEs. The title of these figures tallies the total NFE savings and proportions of instances in receipt of these savings.

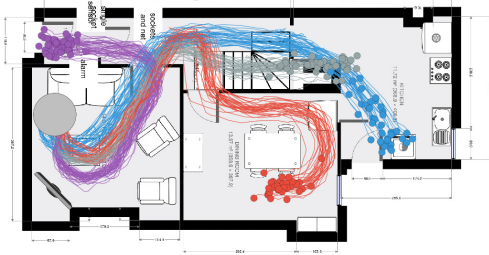
The remaining figures in the left column of fig. 11 present the NFE histograms on three datasets, and baseline, VF+TVLoss and SVFM results are shown in blue, orange and green respectively. The basic NODE models are shown here to require the most function evaluations. Although incorporating TVLoss reduces the number of samples required, the greatest savings in all cases are obtained with the SVFM model since it directly facilitates linear solutions.

The NFE distributions from pairs of models are compared in the right column of fig. 11 on a per-instance basis. Here, the  $(i, j)$ -th component of the matrix counts the number of instances for which the first model made  $i$  evaluations and the second made  $j$ . The NFEs for VF, VF+TVLoss and SVFM models are compared on the Exclusive-Or (XOR) dataset. This figure shows that NFE savings are gained by

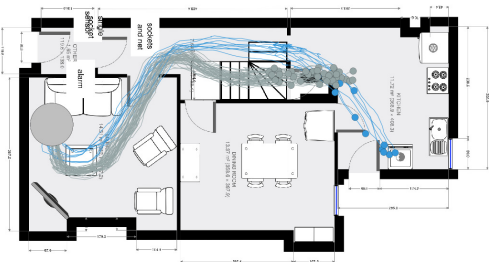
using the proposed method and TVLoss losses over baselines, with the biggest savings delivered by SVFM. In particular with figs. 11(d) and 11(f) we see that the proposed method dominates VFs in all cases. Per-instance NFE distributions on the other datasets were also investigated and, in general, show more significant savings are obtained with the SVFM approach.

#### 4.4 Behavioural forecasting

In behavioural forecasting we are interested in predicting the future location and trajectory of a person given knowledge of their original starting position. SVFMs utility in delivering these forecasts is demonstrated with a bespoke dataset captured in a residential environment. A volunteer was recorded with a Light Detection And Ranging (LiDAR) data collection unit and the resulting pointcloud was processed with Simultaneous Localisation And Mapping (SLAM) techniques to produce relative time and location data. For the purposes of this experiment, data was collected over four consistent paths uniformly that start in the living room and end at the front door, kitchen, landing and dining room. The behavioural data will be released on publication in raw and processed forms.



(a) Forecasts learnt from uniform endpoints.



(b) Forecast based on time of day.

**Figure 12.** Visualisation of behavioural forecasting starting from the couch in the living room (grey circle).

In fig. 12(a) the floor plan of the residential environment and the approximate location of furniture are shown. An SVFM model is learnt on the data above with the forecasting loss (eq. (11)), and 200 paths sampled from the learnt model are overlaid on this figure. Four clear trajectories corresponding to the four endpoints mentioned above can be seen in the figure and are colour coded according to endpoint. Although the essential trajectory of the paths is the same, each particular path is unique. This is due to directional and length uncertainty. Length uncertainty is particularly notable with the kitchen

trajectories (blue) in which endpoints span a region of approximately two metres.

The distribution over endpoints was balanced in our data capture, and hence in fig. 12(a). Although this clearly illustrates the model’s capability in forecasting behaviour, it does not represent the dynamics of naturalistic behaviour since people will not enter rooms uniformly at random in general. In order to demonstrate the model’s ability to capture naturalistic behaviour, we synthesise a dataset where the encountered paths are conditioned on the ToD. ‘Day’ and ‘night’ periods are defined, and during the daytime all paths are walked with uniform probability. During ‘night,’ however, the landing and kitchen are selected with probability 0.9 and 0.1 respectively. An SVFM model influenced by the time tuple is learnt on these paths. We test with the same conditions of fig. 12(a) during the ‘day,’ and endpoints mimic those in fig. 12(a). However, a counterfactual query of the following form was made: ‘what would the endpoints be had the ToD been night instead of day?’ The resulting paths are shown in fig. 12(b), and the model has delivered on the expectation that ‘landing’ endpoints should be favoured given the temporal context of the query. This is a more complete demonstration of the cyclic continuation demonstrated earlier in fig. 10.

Assessment of forecasting models like this are of particular interest in healthcare domains. A growing pressure for passive assessment disease state and prognosis is leading to better characterisation of symptom such as ‘wandering behaviour’ for patients with Alzheimer’s disease. This symptom is characterised by high-entropy transit within the home, and this is exactly what is shown in fig. 12(a). These are easily distinguished from the low-entropy transitions such as those shown in fig. 12(b). Baseline NODE and ANODE models are unable to adequately characterise these behavioural problems due to the natural variation of walked paths in these domains.

## 5 Conclusion

This paper is concerned with encouraging parsimonious advections from neural ordinary differential equation models, and we draw two main conclusions in this work that are fortified by our experiments: 1. loss functions acting on particle trajectories lead to models with lower effective complexity; and 2. introducing uncertainty to vector fields directly delivers simpler solutions. To the best of our knowledge, trajectory-based losses have not previously been explored within the NODE paradigm, and the three that we introduce are shown here to be a straightforward and effective means of controlling model complexity. The most successful model in our experiments is the stochastic vector field mixture that we propose, and it is the vector field mixtures in particular that facilitate improved performance and linearly transported solutions. The utility of these models is demonstrated on illustrative tasks and in human behavioural modelling where we demonstrate its ability to capture the rich variation that is characteristic and forecast human movement. These results in particular suggest that these models may be applied to open behavioural quantification questions in healthcare settings, particularly with Alzheimer’s disease. Future work will build on what is presented here, and in particular will characterise between-resident effects in forecasting and explore behavioural modelling when location is uncertain or latent.

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